SUMMATION OF CERTAIN INFINITE SERIES.

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I note with some surprise that no solution of Problem 221 has yet (November, 1906) reached the MONTHLY. I therefore take the liberty of pointing out the method usually adopted in questions of this type, and embody in my remarks solutions of many similar questions taken from Todhunter and Hogg's Trigonometry, Hobson's Trigonometry (our best English work on the subject), and from various Cambridge Scholarship and other papers of recent years. The whole will, I hope, form a useful summary for those to whom the methods are unfamiliar.

1. To save space, we write the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$ to ∞ in the form $\sum_{n=1}^{n=\infty}\frac{1}{n^2}$; $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$ to ∞, in the form $\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2}$.

The sign $\equiv$ is used for "identical with." We know that

$$\left.\begin{aligned}\frac{\sin\theta}{\theta}&=1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}-\frac{\theta^6}{7!}+\ldots\equiv\sum_{n=1}^{n=\infty}(-1)^{n-1}\frac{\theta^{2n-2}}{(2n-1)!}\\ \text{and also }&=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots\equiv\prod_{n=1}^{n=\infty}\left(1-\frac{\theta^2}{n^2\pi^2}\right)\end{aligned}\right\}(A).$$

Taking logarithms of the two right hand expressions, and expanding in powers of θ, we may equate the coefficients of the respective powers of θ in the system (A).

Thus equating coefficients of θ^2 we have

$$(a)\qquad -\frac{\theta^2}{3!}=-\theta^2\left(\sum_{n=1}^{n=\infty}\frac{1}{n^2\pi^2}\right),\text{ whence }\sum_{n=1}^{n=\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

Hence, if a, b, c, ... denote all the prime numbers except unity,

$$(b)\qquad (1-a^{-2})^{-1}(1-b^{-2})^{-1}(1-c^{-2})^{-1}\ldots$$

$$=(1+a^{-2}+a^{-4}+\ldots)(1+b^{-2}+b^{-4}+\ldots)(1+c^{-2}+c^{-4}+\ldots)\ldots$$

$$=1+(a^{-2}+b^{-2}+c^{-2}+\ldots)+\ldots+(a^{-r}b^{-s}\ldots)^2+\ldots$$

$$=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\ldots\textit{ ad infinitum}=\frac{\pi^2}{6}.$$

We also have at once $\overset{\infty}{\underset{2}{\Pi}}\ (1-a^{-2}) = \frac{6}{\pi^2}$.

Again equating coefficients of θ^4 in (A), we have

$$\frac{1}{5!} - \tfrac{1}{2}\left(\frac{1}{3!}\right)^2 = -\frac{1}{2\pi^4}\sum_{n=1}^{n=\infty}\frac{1}{n^4},$$

and it follows that

(c) $$\sum_{n=1}^{n=\infty}\frac{1}{n^4} = \frac{\pi^4}{12}\left(\tfrac{1}{3}-\tfrac{1}{5}\right) = \frac{\pi^4}{90}.$$

Then, if a, b, c, ... are the prime numbers we have

(d) $$(1-a^{-4})^{-1}(1-b^{-4})^{-1}(1-c^{-4})^{-1}\ldots$$

$$=(1+a^{-4}+a^{-8}+\ldots)(1+b^{-4}+b^{-8}+\ldots)(1+c^{-4}+c^{-8}+\ldots)$$

$$=(1+a^{-4}+b^{-4}+\ldots+(a^r\, b^s\, c^t\ldots)^4+\ldots$$

$$=\sum_{n=1}^{n=\infty}\frac{1}{n^4} = \frac{\pi^4}{90} \text{ by } (c).$$

Thus dividing (d) by (b) we have

(e) $\overset{\infty}{\underset{2}{\Pi}}\ (1+a^{-2}) = \frac{15}{\pi^2}$, which is in No. 221, p. 190. And

$$\overset{\infty}{\underset{2}{\Pi}}\ (1+a^{-2})^{-1} \equiv \frac{2^2}{2^2+1}\cdot\frac{3^2}{3^2+1}\cdot\ \ldots = \frac{\pi^2}{15}.$$

We can connect up the series $\sum_{n=1}^{n=\infty}\frac{1}{n^2}$ with $\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2}$, as follows:

Let $S = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \ldots$

$S_1 = \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \ldots$

Then $S = \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \ldots + \frac{1}{2^n}\left(\frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \ldots\right) = S_1 + \frac{1}{2^n}$.

(f) $\therefore S_1 = \frac{2^n-1}{2^n}S$. So that if S be known, S_1 is known. For instance, if $n=2$, $S=\frac{1}{6}\pi^2$ by (a),

(*g*) and $S_1 \equiv \sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^2} = \frac{2^2-1}{2^2} S = \frac{3}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8}$.

If $n=4$, $S = \frac{\pi^2}{90}$ by (*c*),

(*h*) and $S_1 \equiv \sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^4} = \frac{2^4-1}{2^4} S = \frac{15}{16} \cdot \frac{\pi^4}{90}$ by (*c*) $= \frac{\pi^4}{96}$.

(*i*) Consider $\sum_{n=1}^{n=\infty} (-1)^{n-1} \sum_{n=1}^{n=\infty} \frac{1}{n^2}$.

$$\text{Here } \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

$$= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots - \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$= \frac{\pi^2}{8} - \frac{\pi^2}{24} \text{ by } (g) \text{ and } (a) = \frac{\pi^2}{12}.$$

(*j*) Now consider the identity $\left[\sum_{n=1}^{n=\infty} \frac{1}{n^2}\right]^2 \equiv \sum_{n=1}^{n=\infty} \frac{1}{n^4} + 2 \sum \sum \frac{1}{p^2 q^2}$, where p and q are any of the numbers 1, 2, 3, ..., ∞.

Here $\left(\frac{\pi^2}{6}\right)^2 = \frac{\pi^4}{90} + 2 \sum \sum \frac{1}{q^2 q^2}$, by (*a*) and (*d*).

Then, $\sum \frac{1}{p^2 q^2} = -\frac{\pi^4}{180} + \frac{\pi^4}{72} = \frac{\pi^4}{120}$. Hence the sum of the series formed by multiplying together every two of the terms of the series $\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \ldots$, is $\frac{\pi^4}{120}$.

Now consider the identity,

$$\frac{\frac{1}{2}n.(n+1)}{(2n+1)^4} = \frac{1}{8}\left[\frac{1}{(2n+1)^2} - \frac{1}{(2n+1)^4}\right].$$

$$\text{Here} \sum_{n=1}^{n=\infty} \frac{\frac{1}{2}n(n+1)}{(2n+1)^4} = \frac{1}{8}\left[\sum_{n=1}^{n=\infty} \frac{1}{(2n+1)^2} - \sum_{n=1}^{n=\infty} \frac{1}{(2n+1)^4}\right]$$

$$= \frac{1}{8}\left[\frac{\pi^2}{8} - \frac{\pi^4}{96}\right] \text{ by } (g) \text{ and } (h) = \frac{\pi^2}{64}\left[1 - \frac{\pi^2}{12}\right].$$

[To be continued.]